Linear Transformation based problems

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Q:-1:: find a L.T., $T: R^2 \to R^2$ such that
$T(1,2) = (3,4)$ and $T(0,1) = (0,0)$

Sol:- first of all we show that given vectors of domain of T forms a basis for R^2

i.e To show $(1,2)$ and $(0,1)$ are L.I.

let $\alpha(1,2) + \beta(0,1) = 0$ for any scalar α and β.

$\Rightarrow (\alpha + 0\beta, 2\alpha + \beta) = (0,0)$

$\Rightarrow \boxed{\alpha = 0}$, $2\alpha + \beta = 0$

$0 + \beta = 0$

$\boxed{\beta = 0}$

$\therefore (1,2)$ and $(0,1)$ are L.I.

Now To show $(1,2)$ and $(0,1)$ span $R^2 \to (x,y)$

let $(x,y) = \alpha(1,2) + \beta(0,1)$ ——— (1)

$(x,y) = (\alpha + 0\beta, 2\alpha + \beta)$

$\alpha = x$, $2\alpha + \beta = y$

$2x + \beta = y$

$\beta = y - 2x$

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$\therefore$ equation (1), becomes,

$$(x,y) = x(1,2) + (y-2x)(0,1)$$

$\therefore$ $(1,2)$, $(0,1)$ span R^2.

Now $T(x,y) = T\left[x(1,2) + (y-2x)(0,1)\right]$

$$= x\,T(1,2) + (y-2x)\,T(0,1)$$

$$= x(3,4) + (y-2x)(0,0)$$

$$= (3x, 4x) + (0,0)$$

$$= (3x, 4x)$$

Q:-2:- find $T(x,y,z)$, where $T: R^3 \to R$ is defined by
$T(1,1,1) = 3$, $T(1,1,0) = -4$, $T(1,0,0) = 2$

Sol: first of all we show that given vectors of domain of T forms a basis for R^3 (= domain of T)

To show $(1,1,1)$, $(1,1,0)$. and $(1,0,0)$ are L.I

Consider $\alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,0) = 0$

for any α, β, γ scalars.

$\Rightarrow (\alpha+\beta+\gamma, \ \alpha+\beta, \ \alpha) = (0, 0, 0)$

$\alpha+\beta+\gamma = 0$

$\alpha+\beta = 0$

$\alpha = 0$

Here $\boxed{\alpha=0}$, $\alpha+\beta=0$ $\quad$ $\alpha+\beta+\gamma=0$

$0+\beta=0$ $\quad$ $0+0+\gamma=0$

$\boxed{\beta=0}$ $\quad$ $\boxed{\gamma=0}$

$\therefore (1,1,1), (1,1,0), (1,0,0)$ are L.I.

Now to show $(1,1,1)$, $(1,1,0)$ and $(1,0,0)$ span R^3:

let $(x,y,z) \in R^3$

let $(x,y,z) = \alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,0)$

$(x,y,z) = (\alpha+\beta+\gamma, \ \alpha+\beta, \ \alpha)$

$\therefore \alpha+\beta+\gamma = x$

$\alpha+\beta = y$

$\alpha = z$

$\therefore \boxed{\alpha = z}$ $\qquad \beta = y - \alpha$ $\qquad \alpha + \beta + \gamma = x$

$\boxed{\beta = y - z}$, $\quad y + \gamma = x$

$\boxed{\gamma = x - y}$

Thus $(x, y, z) = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

Hence $(1,1,1)$, $(1,1,0)$ and $(1,0,0)$ span R^3.

$$\therefore T(x,y,z) = z\,T(1,1,1) + (y-z)\,T(1,1,0) + (x-y)\,T(1,0,0)$$
$$= z(3) + (y-z)(-4) + (x-y)(2)$$
$$= 3z - 4y + 4z + 2x - 2y$$
$$= 2x - 6y + 7z$$ which is required Linear Transformation

H.W. find $T(x,y,z)$, where $T: R^3 \to R$ is defined by $T(1,1,1) = 3$, $T(0,1,-2) = 1$, $T(0,0,1) = -2$

Sol:

Hint

1. To show $(1,1,1)$, $(0,1,-2)$, $(0,0,1)$ are L.I.

2. To show $(1,1,1)$, $(0,1,-2)$, $(0,0,1)$ span R^3

3. To find $T(x, y, z)$

Here $T(x,y,z) = 8x - 3y - 2z$ Ans

Q:- Find a linear transformation $T: P_3(x) \to P_2(x)$ such that

$T(1+x) = 1+x, \quad T(2+x) = x + 3x^2, \quad T(x^2) = 0$

Sol: first of all we show that $B = \{1+x, 2+x, x^2\}$ form a basis of $P_3(x)$.

Ist. To Prove B is L.I.

let $\alpha_1, \alpha_2, \alpha_3$ are three scalars such that

$$\alpha_1(1+x) + \alpha_2(2+x) + \alpha_3(x^2) = 0$$

$$(\alpha_1 + 2\alpha_2) + (\alpha_1 + \alpha_2)x + \alpha_3 x^2 = 0 + 0x + 0x^2$$

Equating like powers of x on both sides

$$\alpha_1 + 2\alpha_2 = 0 \quad \text{——} (1)$$

$$\alpha_1 + \alpha_2 = 0 \quad \text{——} (2)$$

$$\alpha_3 = 0 \quad \text{——} (3)$$

from (1) and (2), $\quad \alpha_1 = 0, \quad \alpha_2 = 0$

from (3). $\quad \alpha_3 = 0$

from (3), $\alpha_3 = 0$

$\therefore$ B is L.I set.

2nd part To Prove B Spans $P_3(x)$

let $a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in P_3(x)$

Then $a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \alpha_1 (1+x) + \alpha_2 (2+x) + \alpha_3 x^2$

Equating both sides of like powers of x,

$\therefore$
$$a_0 = \alpha_1 + 2\alpha_2 \quad \text{——} \quad (4)$$
$$a_1 = \alpha_1 + \alpha_2 \quad \text{——} \quad (5)$$
$$a_3 = 0 \quad \text{——} \quad (6)$$
$$\boxed{a_2 = \alpha_3}$$

from (4) and (5), $\alpha_2 = a_0 - a_1$

α_2 using in (5), we get, $a_1 = \alpha_1 + (a_0 - a_1)$
$\Rightarrow \alpha_1 = 2a_1 - a_0$

Thus
$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = (2a_1 - a_0)(1+x) + (a_0 - a_1)(2+x) + a_2 (x^2)$$

$\therefore$ B spans $P_3(x)$.

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Now

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (2a_1 - a_0)\, T(1+x) + (a_0 - a_1)\, T(2+x) + a_2\, T(x^2)$$

$$= (2a_1 - a_0)(1+x) + (a_0 - a_1)(x + 3x^2) + a_0(0)$$

$$= (-a_0 + 2a_1) + a_1 x + 3(a_0 - a_1)x^2$$

which is required linear Transformation

Q:. find a linear transformation $T: R^3 \to R^4$ whose range space spanned by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$

Sol:. The usual basis of R^3 is

$$B = \{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$\therefore$ Range space of T is generated by

$$B_1 = \{T(e_1), T(e_2), T(e_3)\}$$

But it is generated by $\{(1,2,0,-4),$ and

But it is generated by $\{(1,2,0,-4)$, and $(2,0,-1,-3)\}$

$$\therefore \; T(e_1) = (1,2,0,-4)$$

$$T(e_2) = (2,0,-1,-3)$$

$$\text{and } \; T(e_3) = (0,0,0,0)$$

which is not present, we consider as $(0,0,0,0)$.

Now for each $(x,y,z) \in R^3$, we have

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$= xe_1 + ye_2 + ze_3$$

$$\therefore \; T(x,y,z) = T(xe_1 + ye_2 + ze_3)$$

$$= xT(e_1) + yT(e_2) + zT(e_3)$$

$$= x(1,2,0,-4) + y(2,0,-1,+3) + z(0,0,0,0)$$

$$= (x+2y, \; 2x-y, \; -4x-3y)$$

which is required linear transformation.